Exam. Code : 211002
Subject Code : 5540

M.Sc. (Mathematics) 2nd Semester

REAL ANALYSIS—II

Paper—MATH-561

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Attempt any **TWO** questions from each unit. Each question carries equal marks.

UNIT—I

1. State and prove Arzela's theorem.
2. Suppose K is compact and $\{f_n\}$ is a sequence of continuous functions on K and $\{f_n\}$ converges pointwise to a continuous function f on K. Also, $f_n(x) \geq f_{n+1}(x)$, $\forall\ x \in K$, n = 1, 2, 3, ... Then $f_n \to f$ uniformly on K.
3. The sequence of functions $\{f_n\}$ defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies $| f_n(x) - f_m(x) | \leq \varepsilon$.
4. Define equicontinuity. If K is compact and $f_n \in \mathcal{C}(K)$ for n = 1, 2, 3, ... and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then $\{f_n\}$ is uniformly bounded on K and contains a uniformly convergent subsequence.

UNIT—II

5. Define a measurable set. Prove that outer measure of an interval is its length.
6. If m is a countably additive, translation invariant measure defined on a σ-algebra containing the set P, then m[0, 1) is either zero or infinity.
7. If A is countable then show that $m^*A = 0$.
8. Show that the interval (a, ∞) is measurable.

UNIT—III

9. Define a measurable function. Let c be a constant and f and g be two real valued measurable functions defined on the same domain, then $f \pm g$, $f + c$ and cf are also measurable.
10. Define a characteristic function and a simple function. Prove that $\chi_{A \cap B} = \chi_A \cdot \chi_B$ and $\chi_{\bar{A}} = 1 - \chi_A$.
11. Define almost everywhere. If f is measurable function and f = g a.e., then g is measurable.
12. State and prove Egoroff's theorem.

UNIT—IV

13. Give an example of a function which is Lebesgue integrable but not Riemann integrable.
14. State and prove monotone convergence theorem.
15. State and prove bounded convergence theorem.
16. Let f be a non-negative measurable function. Show that $\int f = 0$ implies f = 0 a.e.

UNIT—V

17. State and prove Vitali's lemma.
18. If f is integrable on [a, b] and $\int_a^x f(t)\,dt = 0$, for all $x \in [a, b]$, then f(t) = 0 a.e. in [a, b].
19. Define absolute continuity. Show that every absolutely continuous function is the indefinite integral of its derivative.
20. Let f be an increasing real valued function on the interval [a, b]. Then f is differentiable almost everywhere. The derivative f′ is measurable and $\int_a^b f'(x)\,dx \le f(b) - f(a)$.

Exam. Code : 211002
Subject Code : 5541

M.Sc. (Mathematics) 2nd Semester

TENSORS AND DIFFERENTIAL GEOMETRY

Paper—MATH-562

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Attempt **TWO** questions from each unit. All questions carry equal marks.

UNIT—I

1. Define Certesian Tensor of order 4. Also define contraction and state and prove contraction theorem.
2. Show that δ_{ij} is a tensor of order two.
3. Show that the transformation of a mixed tensor possess the transitive property.
4. Show that Christoffel symbols do not behave like tensor.

UNIT—II

5. Define principal normal and binormal. Find the equations of the principal normal and binormal.
6. State and prove Serret-Frenet formulae.

7. Find the curvature and torsion of the curve

$$x = a(u - \sin u),\ y = a(1 - \cos u),\ z = bu.$$

8. Find the centre and radius of spherical curvature.

UNIT—III

9. Investigate the spherical indicatrices of the circular helix $x = a\cos\theta$, $y = a\sin\theta$, $z = c\theta$, $c \neq 0$.

10. Find the envelop of the plane $lx + my + nz = 0$ where $al^2 + bm^2 + cn^2 = 0$.

11. Find the condition that the surface given by $z = f(x, y)$ may be developable.

12. Calculate the fundamental magnitudes to the surface $2z = ax^2 + 2hxy + by^2$ taking x, y as parameter.

UNIT—IV

13. Define conjugate direction. Find an analytic expression for two directions to be conjugate.

14. Show that the necessary and sufficient condition that the parametric curves be lines of curvature are $F = 0$, $M = 0$.

15. Find the asymptotic lines on the surface $z = x \sin y$.

16. State and prove theorem of Beltrami and Enneper.

UNIT—V

17. Show that the curves $u + v$ = constant are geodesics on the surface with metric

$$(1 + u^2)\,du^2 - 2uv\,du\,dv + (1 + v^2)dv^2.$$

18. Show that geodesic curvature vector of any curve is orthogonal to the curve.

19. State and prove Gauss – Bonnet theorem.

20. Find the condition that surface s may be mapped conformally onto surface s'.

Exam. Code : 211002
Subject Code : 5542

M.Sc. (Mathematics) 2nd Semester

ALGEBRA–II

Paper–MATH–563

Time Allowed—3 Hours] [Maximum Marks—100

Note :– The candidates are required to attempt **two** questions from each unit. Each question carries equal marks.

UNIT–I

1. (a) Prove that an irreducible element in a PID is always prime. 5

   (b) Prove that every Euclidean Domain is a PID. 5

2. (a) Prove that F[x], F field, is an Euclidean ring. 5

   (b) If R is an integral domain with unit element, then prove that any unit in R[x] must be unit in R. 5

3. (a) Is $\mathbb{Z}[x]$ a Principal Ideal Domain ? Justify your answer. 5

   (b) Prove in UFD, two non-zero elements possess HCF. 5

4. Prove that a if a ring R is PID then it is UFD. Is the converse true ? Justify. 10

UNIT–II

5. (a) If $a, b \in K$ are algebraic over F such that $[F(a) : F] = m$ and $[F(b) : F] = n$ and $\gcd(m, n) = 1$, then prove that $[F(a, b) : F] = mn$. 5

(b) Prove that $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right)$ is a simple extension. 5

6. (a) Give an example of an algebraic extension of a field which is not finite. 5

(b) For a prime p, find the degree of splitting field of $x^p - 1$ over $\mathbb{Q}$. 5

7. (a) If K is algebraic over E and E is algebraic over F, then prove that K is algebraic over F. 5

(b) Prove that the splitting field of a polynomial over F is unique upto F-isomorphism. 5

8. (a) Let θ be a root of an irreducible polynomial $x^3 - 2x - 2$. Then find $\frac{1+\theta}{1+\theta+\theta^2}$ in $\mathbb{Q}(\theta)$. 5

(b) Let f(x) be a non-constant polynomial over field F, then prove that there exists an extension E of F in which f(x) has a root. 5

UNIT–III

9. Prove that a regular n-gon is constructible if and only if $\varphi(n)$ is a power of 2. 10

10. (a) Prove that the characteristic of a finite field F is prime number say p and F contains a subfield isomorphic to $\mathbb{Z}_p$. 5

(b) Construct a field with 27 elements. 5

11. (a) Prove that the multiplicative group of non-zero elements of a finite field is cyclic. 5

(b) Show that all the roots of an irreducible polynomial over finite field are distinct. 5

12. Prove that a finite separable extension of a field is simple. 10

UNIT–IV

13. Prove that for a Galois extension E/F, there is 1-1 correspondence between the subgroups of G(E/F) and the subfields of E containing F. 10

14. Let $E = \mathbb{Q}\left(\sqrt[3]{2}, w\right)$, where $w^3 = 1, w \neq 1$. Let $H \subseteq G(E/\mathbb{Q})$ and $H = \{\sigma_1, \sigma_2)$ where σ_1 is the identity map and $\sigma_2\left(\sqrt[3]{2}\right) = \sqrt[3]{2}w$ and $\sigma_2(w) = w^2$. Find E_H. 10

15. Suppose that the Galois group G(E/F) of a polynomial f(x) over F is a solvable group, prove that E is solvable by radicals over F. 10

16. (a) Give an example each of a polynomial which is solvable by radicals and a polynomial which is not solvable by radicals. 7

(b) Express $x_1^3 + x_2^3 + x_3^3$ as a rational function of elementary symmetric function. 3

UNIT–V

17. State fundamental theorem of finitely generated module over PID. Prove that a finitely generated torsion–free module over PID is free. 10

18. State and prove Schur's lemma for simple modules. 10

19. Prove that over PID, a submodule of finitely generated module is finitely generated. Is the result true in general ? Justify. 10

20. Let R be a commutative ring with unity and M, N free R-modules. Prove that $Hom_R(M, N)$ is a free R-module if M is finitely generated. Further if N is also finitely generated, then find the basis of $Hom_R(M, N)$. 10

Exam. Code : 211002
Subject Code : 5543

M.Sc. (Mathematics) 2nd Semester

MECHANICS—II

Paper—MATH–564

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt ten questions in all, selecting **TWO** questions from each unit. All questions carry equal marks.

UNIT—I

I. Prove that :

$$\overrightarrow{V_p} = \overrightarrow{V_0} + \vec{\omega} \times \vec{r}$$

where symbols have usual meaning.

II. Discuss kinematics of rigid lithospheric plate motions on a rotating earth.

III. A force $\vec{F}$ acts on a particle constrained to move along a curve C joining points A and B. Prove that work done is :

$$W = \int_{t_A}^{t_B} A\, dt \text{, A being power.}$$

IV. Find $\vec{I} = m\overrightarrow{v_2} - m\overrightarrow{v_1}$ and then show that if the velocity of a particle of mass m changes from $\overrightarrow{v_1}$ to $\overrightarrow{v_2}$ due to impulse $\vec{I}$, then K.E. gained is $\frac{1}{2}\vec{I}.\left(\overrightarrow{v_2} + \overrightarrow{v_1}\right)$.

UNIT—II

V. Prove that :

$$A^2 \omega_1^2 + B^2 \omega_2^2 + C^2 \omega_3^2 = \text{Const},$$

$\omega_1, \omega_2, \omega_3$ are the angular velocities along and A, B, C are the principal moments of inertia about the axes.

VI. Prove that the motion of a rigid body about a fixed point may be represented by the rolling of an ellipsoid fixed in the body upon a plane fixed in space.

VII. A uniform solid sphere rolls without slipping on a rough horizontal plane which is rotating with uniform angular velocity about a vertical axis. If there are no force acting on the sphere save its weight and the friction at the contact, prove that the focus of the centre of the sphere is a circle.

VIII. A uniform rectangular lamina of sides 2a, 3a is freely hinged to a horizontal axis along one of its shorter edges. This axis is fixed to a vertical shaft which passes through the midpoint of the hinged edge, and the shaft is forced to rotate with constant angular velocity ω. If θ is the inclination of the plate to the downward vertical at time t, show that Euler's dynamical equations can be written as : $a\ddot{\theta} - a\omega^2 \sin\theta\cos\theta = -\frac{1}{2} g \sin\theta$.

UNIT—III

IX. Prove that :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \ (j = 1, 2, ..., n)$$

in usual notations.

X. Discus equilibrium configurations for conservative holonomic dynamical systems.

XI. Determine virtual work function for flyball governor.

XII. Explain normal periods of oscillation.

UNIT—IV

XIII. Prove that the necessary and sufficient condition for $\int_{x_1}^{x_2} f(x, y, y')\,dx$ to be an extremum is that :

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

XIV. Describe Hamilton's principle.

XV. A particle of mass m moves on xy-plane under the influence of a force of alteration to the origin of magnitude $F(r) > 0$, where r is the distance of the mass from the origin. Find the Lagrange's equation of motion.

XVI. Explain extension of the variational method.

UNIT—V

XVII. Find the extremals of the functional

$$\int_0^{\frac{\pi}{2}} \left\{ 2xy + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right\} dt.$$

XVIII. Find the extremals of the functional :

$$I[y(x)] = \int_{x_0}^{x_1} (2xy + (y''')^2)\, dx.$$

XIX. Explain Geodesics.

XX. Explain Galerkin's method.

Exam. Code : 211002
Subject Code : 5544

M.Sc. (Mathematics) 2nd Semester

DIFFERENTIAL AND INTEGRAL EQUATIONS

Paper—MATH-565

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Candidate to attempt **TWO** questions from each unit. Each question carries equal marks.

UNIT—I

1. Prove that the general solution of linear differential equation $Pp + Qq = R$ is of the form $F(u, v) = 0$, where $F(u, v)$ is an arbitrary function of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ which form a solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

2. Find the equation of the integral surface of the differential equation $2y(z - 3)p + (2x - z)q = y(2x - 3)$ which passes through the circle $z = 0$, $x^2 + y^2 = 2x$.

3. Find the surface which is orthogonal to the one parameter system $z = cxy(x^2 + y^2)$ and which passes through the hyperbola $x^2 - y^2 = a^2$, $z = 0$.

4. Use Charpit's method to solve the partial differential equation $(p^2 + q^2)y = qz$.

UNIT—II

5. If f and g are arbitrary functions of their respective arguments, show that $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, provided $\alpha = \sqrt{1 - \frac{v^2}{c^2}}$.

6. Solve the equation :

$$\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}.$$

7. Reduce the following partial differential equation into canonical form and hence solve it :

$$y^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}.$$

8. Solve the wave equation $r = t$ by Monge's method.

UNIT—III

9. Solve the Laplace equation in spherical coordinates by method of separation of variables.

10. The ends A and B of a rod, 10 cm in length are kept at temperature 0°C and 100°C, respectively until the steady state condition prevails. Suddenly the temperature at the end A is increased to 20°C and at the end B is decreased to 60°C. Find the temperature distribution in rod at time t.

11. Obtain the appropriate solution of the radio equation $\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$ appropriate to the case when a periodic e.m.f. $V_0 \cos pt$ is applied at the end $x = 0$ of the line.

12. Solve the following heat conduction equation using Fourier transforms :

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \ -\infty < x < \infty, \ t > 0,$$

subject to the initial and boundary conditions as :

$u(x, 0) = f(x), -\infty < x < \infty$ and $u(x, t) \to 0$ and $\partial u/\partial x \to 0$ as $x \to \infty$.

UNIT—IV

13. Explain the relation between linear non homogeneous differential equation and Volterra integral equation.

14. Explain the method of successive substitution for the solution of Volterra integral equation.

15. Define reciprocal function. If K(x, t) is real and continuous in R, there exists a reciprocal function k(x, t), provided that $M(b - a) < 1$, where M is maximum of K(x, t) in R.

16. Solve the integral equation $u(x) = x + \int_0^x (t - x)\, u(t)dt$.

UNIT—V

17. Solve the Fredholm equation :

$$u(x) = e^x - \frac{e-1}{2} + \frac{1}{2}\int_0^1 u(t)dt.$$

18. Explain the method of successive approximations for the solution of Fredholm integral equation.

19. If K(x, t) is non-zero real and continuous in R and f(x) is non-zero real and continuous I. A function k(x, t) is reciprocal to K(x, t) exists then the Fredholm integral equation $u(x) = f(x) + \int_a^b k(x, t)\, u(t)dt$ has the solution of the form $u(x) = f(x) - \int_a^b K(x, t)\, f(t)dt$.

20. Compute D(λ) for the integral equation :

$$u(x) = f(x) + \lambda \int_0^{\pi} \sin x\, u(t)dt.$$