Exam. Code : 211004
Subject Code : 5565

M.Sc. (Mathematics) 4th Semester

FUNCTIONAL ANALYSIS—II

Paper—MATH-581

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **TWO** questions from each unit. Each question carries equal marks.

UNIT—I

1. Let X and Y be normed spaces, $T \in B(X, Y)$ and (x_n) be a sequence in X. If $x_n \to^w x_0$, then show that $T(x_n) \to^w T(x_0)$.

2. Prove that in a normed space X, strong convergence implies weak convergence with the same limit but converse is not generally true. Justify.

3. Prove that in a finite dimensional normed space X, strong convergence and weak convergence of a sequence (x_n) is equivalent.

4. Let (x_n) be a weakly convergent sequence in a normed linear space, i.e. $x_n \to^w x$. Then show that the weak limit x of x_n is unique and every subsequence of (x_n) converges weakly to x.

UNIT—II

5. Define self adjoint operator and prove the following :

   (a) $(S + T)^* = S^* + T^*$

   (b) $(\alpha T)^* = \alpha T^*, \forall \alpha \in \mathbb{R}$

6. If T is a positive operator on a Hilbert space H, then prove that spectrum $\sigma(T) \subseteq \mathbb{R}^+$.

7. Prove that the adjoint T^* is linear and bounded and $\|T^*\| = \|T\|$.

8. If S and T are normal operator satisfying $ST^* = T^*S$ and $TS^* = S^*T$, show that the sum $S + T$ and product ST are normal.

UNIT—III

9. If $X \neq \{0\}$ is a complex Banach space and $T \in B(X, X)$, then prove that spectrum $\sigma(T) \neq \phi$.

10. Prove that the resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space X is open.

11. Let $A \in B(H)$ be a self adjoint operator, then every eigen values of A are real, where H is a Hilbert space.

12. Prove that all matrices representing a given linear operator $T : X \to X$ on a finite dimensional normed space X relative to various basis for X have the same eigen values.

UNIT—IV

13. Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Then :

    (a) If B is relatively compact then B is totally bounded.

    (b) If B is totally bounded and X is complete, then B is relatively compact.

14. Give an example of a compact operator. Justify your answer.

15. Let $T : X \to X$ be a compact linear operator on a normed space X. If $\dim X = \infty$, then show that $0 \in \sigma(T)$.

16. Let X and Y be normed space and $T : X \to Y$ be a linear operator. Then T is compact iff it maps every bounded sequence (x_n) in X onto a subsequence $(T(x_n))$ in Y which has a convergent subsequence.

UNIT—V

17. Define Banach algebra and give one example each of commutative and non-commutative Banach algebra. Justify your examples.

18. Let A be a complex Banach algebra with identity e. Then for any $x \in A$, prove that the spectrum $\sigma(x)$ is compact and the spectral radius satisfies $\gamma_\sigma(x) \leq \|x\|$.

19. Prove that $\sigma(x) \neq \phi$ for any $x \in A$, where A is a complex Banach algebra with identity e.

20. Let X be a Banach algebra and ϕ is a multiplicative linear functional. Then ϕ is bounded and $\|\phi\| = 1$.

Exam. Code : 211004
Subject Code : 5573

M.Sc. (Mathematics) 4th Semester

NUMBER THEORY

Paper—MATH-586

Time Allowed—3 Hours] [Maximum Marks—100

Note— The candidates are required to attempt two questions from each unit. Each question carries equal marks.

UNIT—I

1. (a) Solve the simultaneous congruence $x \equiv 5 \pmod{11}$, $x \equiv 14 \pmod{29}$, $x \equiv 15 \pmod{31}$. 5

   (b) Find three consecutive integers, each having a square factor. 5

2. State and prove Wilson's Theorem. 10

3. Let the order of a modulo n be h and the order of b modulo n be k. Show that the order of ab modulo n divides hk. In particular, if gcd (h, k) = 1, then the order of ab modulo n is hk. 10

4. Prove that if n has a primitive root then it has exactly $\phi(\phi(n))$ of them. 10

UNIT—II

5. (a) Prove that $x^2 \equiv -1 \pmod p$ has a solution if and only if $p \equiv 1 \pmod 4$. 5

(b) If r is a primitive root of the odd prime p, verify that

$ind_r(-1) = t\ ind_r(p-1) = \frac{p-1}{2}$. 5

6. State and prove Gauss lemma. 10

7. For an odd prime $p \neq 3$, prove that

$$\left(\frac{3}{p}\right) = \begin{cases} 1, \text{if } p \equiv 1, -1 \pmod{12}, \\ -1, \text{if } p \equiv 5, -5 \pmod{12}. \end{cases}$$ 10

8. Let p be an odd prime and gcd (a, p) = 1. Prove that a is quadratic residue of p if and only if $a^{\frac{p-1}{2}} \equiv 1 \pmod p$. Find all quadratic residues of 13. 10

UNIT—III

9. State and prove Mobius inversion formula. 10

10. (a) Prove that $\Sigma_{d/n} \frac{1}{d} = \frac{\sigma(n)}{n}$ for every positive integer n. 5

(b) Prove that $\tau(n)$ is an odd integer if and only if n is a perfect square. 5

11. Find all solutions of the Pythagorean equation $x^2 + y^2 = z^2$ satisfying the conditions gcd (x, y, z) = 1, 2 | x, x, y, z > 0. 10

12. Prove that $x^4 + y^4 = z^4$ has no solution in positive integers. 10

UNIT—IV

13. Prove that any positive integer can be written as sum of fur squares of non-negative integers. 10

14. Let the positive integer n be written as $n = N^2 m$, where m is square free. Then prove that n can be represented as sum of two squares if and only if m contains no prime factor of the form 4k + 3. 10

15. If q is an irrational number, then prove that there are infinitely many rationals $\frac{a}{b}$ such that $|-\theta\frac{a}{b}| < \frac{1}{b^2}$. 10

16. (a) Prove that a positive integer is representable as the difference of two squares if and only if it is the product of two factors that are both odd or both even. 5

(b) Express 113 as sum of two squares. 5

UNIT—V

17. Prove that the value of any finite continued fraction is a rational number. Express $\frac{19}{51}$ as continued fraction. 10

18. Prove that $x^2 - dy^2 = 1$ has no solution if and only if $d \equiv 3 \pmod 4$. Find the least positive solution of $x^2 - 73y^2 = 1$, given that $\sqrt{73} = [8, 1, 1, 5, 5, 1, 1, 16, 1, 1, 5, 5, 1, 1, 16, \ldots]$. 10

19. (a) Prove that for $\theta = [a_0; a_1, a_2 \ldots, a_n]$, $a_0 = [\theta]$ and if $\theta_1 = [a_1; a_2 \ldots, a_n]$, then $\theta = a_0 + \frac{1}{\theta_1}$. 7

(b) Evaluate $[1, 2, 1, 2, \ldots\ldots]$. 3

20. (a) Solve $18x + 5y = 24$ by means of continued fraction. 5

(b) Find the continued fraction form of $\sqrt{3}$. 5

30/5/18 MoP

Exam. Code : 211004
Subject Code : 5575

M.Sc. (Mathematics) 4th Semester

OPERATIONS RESEARCH—II

Paper—MATH–588

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **FIVE** questions in all, selecting **TWO** questions from each Unit. All questions carry equal marks.

UNIT—I

1. If the number of arrivals in some time interval follows a Poisson distribution; show that the distribution of the time interval between successive arrivals is exponential.
2. Explain the basic queueing process. What are the important random variables in queuing system to be investigated ?
3. Explain (M|M|1) : (∞|FCFS) queuing model and obtain the probability density function of waiting time distribution.
4. There is congestion on the platform of a railway station. The trains arrive at the rate of 30 trains per day. The waiting time for any train to hump is exponentially distributed with an average of 36 minutes. Calculate the following :
   (a) The mean queue size
   (b) The probability that queue size exceeds 9.

UNIT—II

5. Obtain the steady state solution of the queuing model (M|M|l) : (N|FCFS).

6. What is multi-channel queuing problem ? Obtain the steady state distributions for the number of units in the system and of waiting time in the queue for the model (M|M|C) : (∞|FCFS).

7. A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean '4' minutes, and if people arrive in a Poisson fashion at the rate of 10 an hour :

   (a) What is the probability of having to wait for service ?

   (b) What is the expected percentage of idle time for each girl ?

8. Explain power supply model and obtain its steady state solution.

UNIT—III

9. What is inventory management ? Briefly explain the major decisions concerning inventory.

10. Obtain the EOQ with finite rate of replenishment and the minimum cost.

11. Discuss various inventory costs. Derive EOQ model for an inventory problem when shortages of costs are allowed.

12. Formulate and solve a mathematical model for all units discounts when shortages are not allowed to obtain optimal value of the order quantity.

UNIT—IV

13. What is replacement problem ? Describe some important replacement situations.

14. Explain how the theory of replacement is used in the problem of replacement of items that fail completely.

15. A research team is planned to raise to a strength of 50 Chemists and then to remain at that level. The wastage of recruits depends on their length of service which is as follows :

Year	1	2	3	4	5	6	7	8	9	10
Total % who have left upon the end of year	5	36	56	63	68	73	79	87	97	100

What is the recruitment per year necessary to maintain the required strength ? There are 8 senior posts for which the length of service is the main criterion. What is the average length of service after which new entrant expects promotion to one of the posts ?

16. What is renewal function ? Show that the renewal rate of a machine is asymptotically reciprocal of the mean life of the machine.

UNIT—V

17. Discuss the methodology of simulation. Explain with illustrations, how event-type simulation is useful in operations research ?

18. Vehicles arrive at a toll gate according to a Poisson distribution with mean arrival rate of 10 vehicles per minute. Each gate can process an arrival in 15 seconds on average. Processing times are normally distributed with $\sigma = 2$ seconds :

    (a) Use Monte Carlo simulation to estimate the number of vehicles waiting in line after 10 minutes of operation with only one toll gate open.

    (b) Use Monte Carlo method to determine the minimum number of toll gates necessary, if it is desired that little or no waiting be required.

19. Discuss the various applications of simulation in detail with illustrations.

20. Using random numbers to simulate a sample, find the probability that a packet of 6 products does not contain any defective product, when the production line produces 10% defective products. Compare your answer with the expected probability.