Exam. Code : 211002
Subject Code : 5541

M.Sc. (Mathematics) 2nd Semester

TENSORS AND DIFFERENTIAL GEOMETRY

Paper—MATH-562

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Attempt **TWO** questions from each unit. All questions carry equal marks.

UNIT—I

1. Define Certesian Tensor of order 4. Also define contraction and state and prove contraction theorem.
2. Show that δ_{ij} is a tensor of order two.
3. Show that the transformation of a mixed tensor possess the transitive property.
4. Show that Christoffel symbols do not behave like tensor.

UNIT—II

5. Define principal normal and binormal. Find the equations of the principal normal and binormal.
6. State and prove Serret-Frenet formulae.

7. Find the curvature and torsion of the curve

$$x = a(u - \sin u),\ y = a(1 - \cos u),\ z = bu.$$

8. Find the centre and radius of spherical curvature.

UNIT—III

9. Investigate the spherical indicatrices of the circular helix $x = a \cos \theta$, $y = a \sin \theta$, $z = c\theta$, $c \neq 0$.

10. Find the envelop of the plane $lx + my + nz = 0$ where $al^2 + bm^2 + cn^2 = 0$.

11. Find the condition that the surface given by $z = f(x, y)$ may be developable.

12. Calculate the fundamental magnitudes to the surface $2z = ax^2 + 2hxy + by^2$ taking x, y as parameter.

UNIT—IV

13. Define conjugate direction. Find an analytic expression for two directions to be conjugate.

14. Show that the necessary and sufficient condition that the parametric curves be lines of curvature are $F = 0$, $M = 0$.

15. Find the asymptotic lines on the surface $z = x \sin y$.

16. State and prove theorem of Beltrami and Enneper.

UNIT—V

17. Show that the curves u + v = constant are geodesics on the surface with metric

$$(1 + u^2)\,du^2 - 2uv\,du\,dv + (1 + v^2)dv^2.$$

18. Show that geodesic curvature vector of any curve is orthogonal to the curve.

19. State and prove Gauss – Bonnet theorem.

20. Find the condition that surface s may be mapped conformally onto surface s'.

Exam. Code : 211002
Subject Code : 5542

M.Sc. (Mathematics) 2nd Semester

ALGEBRA–II

Paper–MATH–563

Time Allowed—3 Hours] [Maximum Marks—100

Note :– The candidates are required to attempt **two** questions from each unit. Each question carries equal marks.

UNIT–I

1. (a) Prove that an irreducible element in a PID is always prime. 5

   (b) Prove that every Euclidean Domain is a PID. 5

2. (a) Prove that F[x], F field, is an Euclidean ring. 5

   (b) If R is an integral domain with unit element, then prove that any unit in R[x] must be unit in R. 5

3. (a) Is $\mathbb{Z}[x]$ a Principal Ideal Domain ? Justify your answer. 5

   (b) Prove in UFD, two non-zero elements possess HCF. 5

4. Prove that a if a ring R is PID then it is UFD. Is the converse true ? Justify. 10

UNIT–II

5. (a) If $a, b \in K$ are algebraic over F such that $[F(a) : F] = m$ and $[F(b) : F] = n$ and $\gcd(m, n) = 1$, then prove that $[F(a, b) : F] = mn$. 5

(b) Prove that $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{5}\right)$ is a simple extension. 5

6. (a) Give an example of an algebraic extension of a field which is not finite. 5

(b) For a prime p, find the degree of splitting field of $x^p - 1$ over $\mathbb{Q}$. 5

7. (a) If K is algebraic over E and E is algebraic over F, then prove that K is algebraic over F. 5

(b) Prove that the splitting field of a polynomial over F is unique upto F-isomorphism. 5

8. (a) Let θ be a root of an irreducible polynomial $x^3 - 2x - 2$. Then find $\dfrac{1+\theta}{1+\theta+\theta^2}$ in $\mathbb{Q}(\theta)$. 5

(b) Let f(x) be a non-constant polynomial over field F, then prove that there exists an extension E of F in which f(x) has a root. 5

UNIT–III

9. Prove that a regular n-gon is constructible if and only if $\varphi(n)$ is a power of 2. 10

10. (a) Prove that the characteristic of a finite field F is prime number say p and F contains a subfield isomorphic to $\mathbb{Z}_p$. 5

(b) Construct a field with 27 elements. 5

11. (a) Prove that the multiplicative group of non-zero elements of a finite field is cyclic. 5

(b) Show that all the roots of an irreducible polynomial over finite field are distinct. 5

12. Prove that a finite separable extension of a field is simple. 10

UNIT–IV

13. Prove that for a Galois extension E/F, there is 1-1 correspondence between the subgroups of G(E/F) and the subfields of E containing F. 10

14. Let $E = \mathbb{Q}\left(\sqrt[3]{2}, w\right)$, where $w^3 = 1, w \neq 1$. Let $H \subseteq G(E/\mathbb{Q})$ and $H = \{\sigma_1, \sigma_2)$ where σ_1 is the identity map and $\sigma_2\left(\sqrt[3]{2}\right) = \sqrt[3]{2}w$ and $\sigma_2(w) = w^2$. Find E_H. 10

15. Suppose that the Galois group G(E/F) of a polynomial f(x) over F is a solvable group, prove that E is solvable by radicals over F. 10

16. (a) Give an example each of a polynomial which is solvable by radicals and a polynomial which is not solvable by radicals. 7

(b) Express $x_1^3 + x_2^3 + x_3^3$ as a rational function of elementary symmetric function. 3

UNIT–V

17. State fundamental theorem of finitely generated module over PID. Prove that a finitely generated torsion–free module over PID is free. 10

18. State and prove Schur's lemma for simple modules. 10

19. Prove that over PID, a submodule of finitely generated module is finitely generated. Is the result true in general ? Justify. 10

20. Let R be a commutative ring with unity and M, N free R-modules. Prove that $Hom_R(M, N)$ is a free R-module if M is finitely generated. Further if N is also finitely generated, then find the basis of $Hom_R(M, N)$. 10

Exam. Code : 211002
Subject Code : 5543

M.Sc. (Mathematics) 2nd Semester

MECHANICS—II

Paper—MATH–564

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt ten questions in all, selecting **TWO** questions from each unit. All questions carry equal marks.

UNIT—I

I. Prove that :

$$\overrightarrow{V_p} = \overrightarrow{V_0} + \vec{\omega} \times \vec{r}$$

where symbols have usual meaning.

II. Discuss kinematics of rigid lithospheric plate motions on a rotating earth.

III. A force $\vec{F}$ acts on a particle constrained to move along a curve C joining points A and B. Prove that work done is :

$$W = \int_{t_A}^{t_B} A\, dt, \text{ A being power.}$$

IV. Find $\vec{I} = m\overrightarrow{v_2} - m\overrightarrow{v_1}$ and then show that if the velocity of a particle of mass m changes from $\overrightarrow{v_1}$ to $\overrightarrow{v_2}$ due to impulse $\vec{I}$, then K.E. gained is $\frac{1}{2}\vec{I}.\left(\overrightarrow{v_2} + \overrightarrow{v_1}\right)$.

UNIT—II

V. Prove that :

$$A^2 \omega_1^2 + B^2 \omega_2^2 + C^2 \omega_3^2 = \text{Const},$$

$\omega_1, \omega_2, \omega_3$ are the angular velocities along and A, B, C are the principal moments of inertia about the axes.

VI. Prove that the motion of a rigid body about a fixed point may be represented by the rolling of an ellipsoid fixed in the body upon a plane fixed in space.

VII. A uniform solid sphere rolls without slipping on a rough horizontal plane which is rotating with uniform angular velocity about a vertical axis. If there are no force acting on the sphere save its weight and the friction at the contact, prove that the focus of the centre of the sphere is a circle.

VIII. A uniform rectangular lamina of sides 2a, 3a is freely hinged to a horizontal axis along one of its shorter edges. This axis is fixed to a vertical shaft which passes through the midpoint of the hinged edge, and the shaft is forced to rotate with constant angular velocity ω. If θ is the inclination of the plate to the downward vertical at time t, show that Euler's dynamical equations can be written as : $a\ddot{\theta} - a\,\omega^2 \sin\theta\cos\theta = -\frac{1}{2} g \sin\theta$.

UNIT—III

IX. Prove that :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \ (j = 1, 2, ..., n)$$

in usual notations.

X. Discus equilibrium configurations for conservative holonomic dynamical systems.

XI. Determine virtual work function for flyball governor.

XII. Explain normal periods of oscillation.

UNIT—IV

XIII. Prove that the necessary and sufficient condition for $\int_{x_1}^{x_2} f(x, y, y')\,dx$ to be an extremum is that :

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

XIV. Describe Hamilton's principle.

XV. A particle of mass m moves on xy-plane under the influence of a force of alteration to the origin of magnitude $F(r) > 0$, where r is the distance of the mass from the origin. Find the Lagrange's equation of motion.

XVI. Explain extension of the variational method.

UNIT—V

XVII. Find the extremals of the functional

$$\int_0^{\frac{\pi}{2}} \left\{ 2xy + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right\} dt .$$

XVIII. Find the extremals of the functional :

$$I[y(x)] = \int_{x_0}^{x_1} (2xy + (y''')^2)\, dx .$$

XIX. Explain Geodesics.

XX. Explain Galerkin's method.

Exam. Code : 211002
Subject Code : 5544

M.Sc. (Mathematics) 2nd Semester

DIFFERENTIAL AND INTEGRAL EQUATIONS

Paper—MATH-565

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Candidate to attempt **TWO** questions from each unit. Each question carries equal marks.

UNIT—I

1. Prove that the general solution of linear differential equation $Pp + Qq = R$ is of the form $F(u, v) = 0$, where $F(u, v)$ is an arbitrary function of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ which form a solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

2. Find the equation of the integral surface of the differential equation $2y(z - 3)p + (2x - z)q = y(2x - 3)$ which passes through the circle $z = 0$, $x^2 + y^2 = 2x$.

3. Find the surface which is orthogonal to the one parameter system $z = cxy(x^2 + y^2)$ and which passes through the hyperbola $x^2 - y^2 = a^2$, $z = 0$.

4. Use Charpit's method to solve the partial differential equation $(p^2 + q^2)y = qz$.

UNIT—II

5. If f and g are arbitrary functions of their respective arguments, show that $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$, provided $\alpha = \sqrt{1 - \frac{v^2}{c^2}}$.

6. Solve the equation :

$$\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}.$$

7. Reduce the following partial differential equation into canonical form and hence solve it :

$$y^2\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}.$$

8. Solve the wave equation r = t by Monge's method.

UNIT—III

9. Solve the Laplace equation in spherical coordinates by method of separation of variables.

10. The ends A and B of a rod, 10 cm in length are kept at temperature 0°C and 100°C, respectively until the steady state condition prevails. Suddenly the temperature at the end A is increased to 20°C and at the end B is decreased to 60°C. Find the temperature distribution in rod at time t.

11. Obtain the appropriate solution of the radio equation $\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$ appropriate to the case when a periodic e.m.f. $V_0 \cos pt$ is applied at the end x = 0 of the line.

12. Solve the following heat conduction equation using Fourier transforms :

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \; -\infty < x < \infty, \; t > 0,$$

subject to the initial and boundary conditions as :

$u(x, 0) = f(x), -\infty < x < \infty$ and $u(x, t) \to 0$ and $\partial u/\partial x \to 0$ as $x \to \infty$.

UNIT—IV

13. Explain the relation between linear non homogeneous differential equation and Volterra integral equation.

14. Explain the method of successive substitution for the solution of Volterra integral equation.

15. Define reciprocal function. If K(x, t) is real and continuous in R, there exists a reciprocal function k(x, t), provided that M(b – a) < 1, where M is maximum of K(x, t) in R.

16. Solve the integral equation $u(x) = x + \int_0^x (t - x)\, u(t)\,dt$.

UNIT—V

17. Solve the Fredholm equation :

$$u(x) = e^x - \frac{e-1}{2} + \frac{1}{2}\int_0^1 u(t)dt.$$

18. Explain the method of successive approximations for the solution of Fredholm integral equation.

19. If K(x, t) is non-zero real and continuous in R and f(x) is non-zero real and continuous I. A function k(x, t) is reciprocal to K(x, t) exists then the Fredholm integral equation $u(x) = f(x) + \int_a^b k(x, t)\, u(t)dt$ has the solution of the form $u(x) = f(x) - \int_a^b K(x, t)\, f(t)dt$.

20. Compute D(λ) for the integral equation :

$$u(x) = f(x) + \lambda \int_0^{\pi} \sin x\, u(t)dt.$$

Exam. Code : 211004
Subject Code : 5565

M.Sc. (Mathematics) 4th Semester

FUNCTIONAL ANALYSIS—II

Paper—MATH–581

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **TWO** questions from each unit. Each question carries equal marks.

UNIT—I

1. Let X and Y be normed spaces, $T \in B(X, Y)$ and (x_n) be a sequence in X. If $x_n \rightarrow^w x_0$, then show that $T(x_n) \rightarrow^w T(x_0)$.

2. Prove that in a normed space X, strong convergence implies weak convergence with the same limit but converse is not generally true. Justify.

3. Prove that in a finite dimensional normed space X, strong convergence and weak convergence of a sequence (x_n) is equivalent.

4. Let (x_n) be a weakly convergent sequence in a normed linear space, i.e. $x_n \rightarrow^w x$. Then show that the weak limit x of x_n is unique and every subsequence of (x_n) converges weakly to x.

UNIT—II

5. Define self adjoint operator and prove the following :

   (a) $(S + T)^* = S^* + T^*$

   (b) $(\alpha T)^* = \alpha T^*$, $\forall \alpha \in \mathbb{R}$

6. If T is a positive operator on a Hilbert space H, then prove that spectrum $\sigma(T) \subseteq \mathbb{R}^+$.

7. Prove that the adjoint T^* is linear and bounded and $\|T^*\| = \|T\|$.

8. If S and T are normal operator satisfying $ST^* = T^*S$ and $TS^* = S^*T$, show that the sum S + T and product ST are normal.

UNIT—III

9. If $X \neq \{0\}$ is a complex Banach space and $T \in B(X, X)$, then prove that spectrum $\sigma(T) \neq \phi$.

10. Prove that the resolvent set $\rho(T)$ of a bounded linear operator T on a complex Banach space X is open.

11. Let $A \in B(H)$ be a self adjoint operator, then every eigen values of A are real, where H is a Hilbert space.

12. Prove that all matrices representing a given linear operator $T : X \to X$ on a finite dimensional normed space X relative to various basis for X have the same eigen values.

UNIT—IV

13. Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Then :

    (a) If B is relatively compact then B is totally bounded.

    (b) If B is totally bounded and X is complete, then B is relatively compact.

14. Give an example of a compact operator. Justify your answer.

15. Let $T : X \to X$ be a compact linear operator on a normed space X. If $\dim X = \infty$, then show that $0 \in \sigma(T)$.

16. Let X and Y be normed space and $T : X \to Y$ be a linear operator. Then T is compact iff it maps every bounded sequence (x_n) in X onto a subsequence $(T(x_n))$ in Y which has a convergent subsequence.

UNIT—V

17. Define Banach algebra and give one example each of commutative and non-commutative Banach algebra. Justify your examples.

18. Let A be a complex Banach algebra with identity e. Then for any $x \in A$, prove that the spectrum $\sigma(x)$ is compact and the spectral radius satisfies $\gamma_\sigma(x) \leq \|x\|$.

19. Prove that $\sigma(x) \neq \phi$ for any $x \in A$, where A is a complex Banach algebra with identity e.

20. Let X be a Banach algebra and ϕ is a multiplicative linear functional. Then ϕ is bounded and $\|\phi\| = 1$.

Exam. Code : 211004
Subject Code : 5573

M.Sc. (Mathematics) 4th Semester

NUMBER THEORY

Paper—MATH-586

Time Allowed—3 Hours] [Maximum Marks—100

Note— The candidates are required to attempt two questions from each unit. Each question carries equal marks.

UNIT—I

1. (a) Solve the simultaneous congruence $x \equiv 5 \pmod{11}$, $x \equiv 14 \pmod{29}$, $x \equiv 15 \pmod{31}$. 5

   (b) Find three consecutive integers, each having a square factor. 5

2. State and prove Wilson's Theorem. 10

3. Let the order of a modulo n be h and the order of b modulo n be k. Show that the order of ab modulo n divides hk. In particular, if gcd (h, k) = 1, then the order of ab modulo n is hk. 10

4. Prove that if n has a primitive root then it has exactly $\phi(\phi(n))$ of them. 10

UNIT—II

5\. (a) Prove that $x^2 \equiv -1 \pmod p$ has a solution if and only if $p \equiv 1 \pmod 4$. 5

(b) If r is a primitive root of the odd prime p, verify that

$ind_r(-1) = t\ ind_r(p-1) = \frac{p-1}{2}$. 5

6\. State and prove Gauss lemma. 10

7\. For an odd prime $p \neq 3$, prove that

$$\left(\frac{3}{p}\right) = \begin{cases} 1, \text{if } p \equiv 1, -1 \pmod{12}, \\ -1, \text{if } p \equiv 5, -5 \pmod{12}. \end{cases}$$ 10

8\. Let p be an odd prime and gcd (a, p) = 1. Prove that a is quadratic residue of p if and only if $a^{\frac{p-1}{2}} \equiv 1 \pmod p$. Find all quadratic residues of 13. 10

UNIT—III

9\. State and prove Mobius inversion formula. 10

10\. (a) Prove that $\Sigma_{d/n} \frac{1}{d} = \frac{\sigma(n)}{n}$ for every positive integer n. 5

(b) Prove that $\tau(n)$ is an odd integer if and only if n is a perfect square. 5

11\. Find all solutions of the Pythagorean equation $x^2 + y^2 = z^2$ satisfying the conditions gcd (x, y, z) = 1, 2 | x, x, y, z > 0. 10

12\. Prove that $x^4 + y^4 = z^4$ has no solution in positive integers. 10

UNIT—IV

13\. Prove that any positive integer can be written as sum of fur squares of non-negative integers. 10

14\. Let the positive integer n be written as $n = N^2 m$, where m is square free. Then prove that n can be represented as sum of two squares if and only if m contains no prime factor of the form 4k + 3. 10

15\. If q is an irrational number, then prove that there are infinitely many rationals $\frac{a}{b}$ such that $|-\theta\frac{a}{b}| < \frac{1}{b^2}$. 10

16\. (a) Prove that a positive integer is representable as the difference of two squares if and only if it is the product of two factors that are both odd or both even. 5

(b) Express 113 as sum of two squares. 5

UNIT—V

17\. Prove that the value of any finite continued fraction is a rational number. Express $\frac{19}{51}$ as continued fraction. 10

18. Prove that $x^2 - dy^2 = 1$ has no solution if and only if $d \equiv 3 \pmod 4$. Find the least positive solution of $x^2 - 73y^2 = 1$, given that $\sqrt{73} = [8, 1,1, 5, 5, 1, 1, 16, 1,1, 5, 5, 1, 1, 16, \ldots]$. 10

19. (a) Prove that for $\theta = [a_0; a_1, a_2 \ldots, a_n]$, $a_0 = [\theta]$ and if $\theta_1 = [a_1; a_2 \ldots, a_n]$, then $\theta = a_0 + \frac{1}{\theta_1}$. 7

(b) Evaluate $[1, 2, 1, 2, \ldots\ldots]$. 3

20. (a) Solve $18x + 5y = 24$ by means of continued fraction. 5

(b) Find the continued fraction form of $\sqrt{3}$. 5

30/5/18 MOP

Exam. Code : 211004
Subject Code : 5575

M.Sc. (Mathematics) 4th Semester

OPERATIONS RESEARCH—II

Paper—MATH–588

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **FIVE** questions in all, selecting **TWO** questions from each Unit. All questions carry equal marks.

UNIT—I

1. If the number of arrivals in some time interval follows a Poisson distribution; show that the distribution of the time interval between successive arrivals is exponential.

2. Explain the basic queueing process. What are the important random variables in queuing system to be investigated ?

3. Explain (M|M|1) : (∞|FCFS) queuing model and obtain the probability density function of waiting time distribution.

4. There is congestion on the platform of a railway station. The trains arrive at the rate of 30 trains per day. The waiting time for any train to hump is exponentially distributed with an average of 36 minutes. Calculate the following :

   (a) The mean queue size

   (b) The probability that queue size exceeds 9.

UNIT—II

5. Obtain the steady state solution of the queuing model (M|M|l) : (N|FCFS).

6. What is multi-channel queuing problem ? Obtain the steady state distributions for the number of units in the system and of waiting time in the queue for the model (M|M|C) : (∞|FCFS).

7. A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean '4' minutes, and if people arrive in a Poisson fashion at the rate of 10 an hour :

   (a) What is the probability of having to wait for service ?

   (b) What is the expected percentage of idle time for each girl ?

8. Explain power supply model and obtain its steady state solution.

UNIT—III

9. What is inventory management ? Briefly explain the major decisions concerning inventory.

10. Obtain the EOQ with finite rate of replenishment and the minimum cost.

11. Discuss various inventory costs. Derive EOQ model for an inventory problem when shortages of costs are allowed.

12. Formulate and solve a mathematical model for all units discounts when shortages are not allowed to obtain optimal value of the order quantity.

UNIT—IV

13. What is replacement problem ? Describe some important replacement situations.

14. Explain how the theory of replacement is used in the problem of replacement of items that fail completely.

15. A research team is planned to raise to a strength of 50 Chemists and then to remain at that level. The wastage of recruits depends on their length of service which is as follows :

Year	1	2	3	4	5	6	7	8	9	10
Total % who have left upon the end of year	5	36	56	63	68	73	79	87	97	100

What is the recruitment per year necessary to maintain the required strength ? There are 8 senior posts for which the length of service is the main criterion. What is the average length of service after which new entrant expects promotion to one of the posts ?

16. What is renewal function ? Show that the renewal rate of a machine is asymptotically reciprocal of the mean life of the machine.

UNIT—V

17. Discuss the methodology of simulation. Explain with illustrations, how event-type simulation is useful in operations research ?

18. Vehicles arrive at a toll gate according to a Poisson distribution with mean arrival rate of 10 vehicles per minute. Each gate can process an arrival in 15 seconds on average. Processing times are normally distributed with $\sigma = 2$ seconds :

    (a) Use Monte Carlo simulation to estimate the number of vehicles waiting in line after 10 minutes of operation with only one toll gate open.

    (b) Use Monte Carlo method to determine the minimum number of toll gates necessary, if it is desired that little or no waiting be required.

19. Discuss the various applications of simulation in detail with illustrations.

20. Using random numbers to simulate a sample, find the probability that a packet of 6 products does not contain any defective product, when the production line produces 10% defective products. Compare your answer with the expected probability.