23. 8/12

Exam. Code : 211001
Subject Code : 5472

M.Sc. Mathematics Ist Semester

REAL ANALYSIS–I

Paper–MATH-551

Time Allowed—3 Hours] [Maximum Marks—100

Note :– Attempt **two** questions from each Unit. All questions carry **10** marks each.

UNIT–I

1. What is an open sphere in a metric space ? What are the open spheres in discrete metric space ? Prove that collection of all arbitrary union of open spheres is closed under finite intersections. Also give an example of two different metrics on a set for which the collection of all arbitrary unions of open spheres is same.

2. Prove that there cannot be any surjection from the set of integers to the set of all subsets of integers.

3. Prove that the only compact subsets of the real line are closed and bounded.

4. Prove that for any two disjoint compact sets A and B in a metric space, there exists two disjoint open sets U and V such that A is contained in U and B is contained in V.

UNIT–II

5. Prove that any set contained between a connected set and its closure is also connected.

(Contd.)

6. Let $\{A_n \mid n \in \mathbf{Z}^+\}$ be a countable collection of sets such that each A_n is connected and for each n, $A_n \cap A_{n+1}$ is non empty. Then prove that $\cup_n A_n$ is connected. Prove all the results that you use.

7. Prove that in a metric space, components of open sets are open if and only if every open set is a union of connected open sets.

8. Prove that every function of bounded variation is a difference of two bounded monotonic functions.

UNIT–III

9. State and prove the Cantors Intersection Theorem.

10. Prove that every metric space is a dense subspace of a complete metric space.

11. State and prove Banach's Contraction Principle.

12. State and prove a necessary and sufficient criteria for a metric space to be complete.

UNIT–IV

13. Prove that continuous image of a connected set is connected and continuous image of a compact set is compact.

14. Prove that a map $f : X \to Y$ between metric spaces is continuous at each point of X if and only if inverse image of each open subset of Y is open in X.

15. Let $f : A \to Y$ and $g : B \to Y$ be continuous maps, where $X = A \cup B$ such that A and B are disjoint and both A and B are either open in X or are both closed in X. Then prove that there is a continuous map $h : X \to Y$ such that $h \mid A = f$ and $h \mid B = g$.

16. Prove that every continuous function between metric spaces is uniformly continuous.

UNIT–V

17. State and prove a sufficient condition for the existence of the Riemann-Stieltjes Integral.

18. Prove that if f is Riemann Steiltjes integrable on [a, b] and $\int f dh = 0$ for every monotonic f then h is a constant function on [a, b].

19. State and prove the fundamental theorem of calculus.

20. State and prove the second mean value theorem for the Reimann-Stieltjes integral.

Exam. Code : 211001
Subject Code : 5473

M.Sc. Mathematics 1st Semester

COMPLEX ANALYSIS

Paper—MATH-552

Time Allowed—3 Hours] [Maximum Marks—100

Note :—Attempt **TWO** questions from each unit. All questions carry equal marks.

UNIT—I

1. Show that continuity is a necessary but not a sufficient condition for the existence of a finite derivative.

2. Examine the nature of the function

$$f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}}, z \neq 0, f(0) = 0$$

in the region including the origin.

3. An electrostatic field in xy – Plane is given by the potential function $\phi = 3x^2 y - y^3$, find the stream function.

4. If $w = f(z)$ is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log | f'(z) | = 0$$

If $| f'(z) |$ is the product of a function of x and function of y, show that $f'(z) = \exp(\alpha z^2 + \beta z + \gamma)$ where α is real and β and γ are complex constants.

UNIT—II

5. Define complex line integral and evaluate $\int_{-i}^{i} |z|\,dz$ along the right half of the unit circle $|z| = 1$ described in the counter — clockwise direction.

6. State and prove Cauchy's integral theorem.

7. If $f|z|$ is analytic in a region including the circle $|z| \le R$, prove that for $0 < r < R$

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(R e^{i\varphi})\, d\varphi}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}.$$

where $a = r e^{i\theta}$ is any point of the domain $|z| < R$.

8. State and prove Liouville's theorem.

UNIT—III

9. State Laurent's theorem and prove its Uniqueness.

10. State and prove minimum modulus principle.

11. If all the zeros of a polynomial lie in a half plane. Then all the zeros of derivative also lie in the same half plane.

12. State Argument principle. Use Rouche's theorem to find the number of zeros of the polynomial $2z^4 - 2z^3 + z^2 + 11$ inside the circle $|z| = 1$.

UNIT—IV

13. Define residue of a function $f|z|$ at $z = a$. Find the residue of $z^3 / (z-1)^4 (z-2)(z-3)$ at the poles of the function.

14. State and prove Cauchy's residue theorem.

15. Evaluate $\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta$.

16. Prove that if $0 < a < 1$, then

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\, dn = \frac{\pi}{\sin a\pi}.$$

Exam. Code : 211001
Subject Code : 5474

M.Sc. (Mathematics) Ist Semester

ALGEBRA—I

Paper—MATH-553

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **TWO** questions from each Unit. Each question carries equal marks.

UNIT—I

1. Prove that a finite semi-group G is a group if and only if G satisfies both cancellation laws.

2. State and prove Lagrange's Theorem.

3. If p is the smallest prime factor of the order of a finite group G, prove that any subgroup of index p is normal in G.

4. (a) Find all the subgroups of $\mathbb{Z}/21\ \mathbb{Z}$.

   (b) If H is a subgroup of a group G such that $x^2 \in H\ \forall\ x \in G$ then show that H is normal subgroup of G.

UNIT—II

5. (a) Show that for $G = S_3$ then G', commutator subgroup of G, is A_3.

   (b) Let G be a group of order 231, show that Sylow 11-subgroup of G is contained in Z(G), centre of G.

6. State and prove Fundamental Theorem of Homomorphism for groups.

7. (a) Let G be a group such that G/Z(G) is cyclic show that G is abelian.

(b) Show that a cyclic group of order 8 is homomorphic to a cyclic group of order 4.

8. (a) For any group G, prove that $In(G) \cong G/Z(G)$.

(b) Prove that the group of automorphisms of a cyclic group is abelian.

UNIT—III

9. (a) Prove that any two disjoint permutations commute.

(b) Show that A_4 is the only subgroup of order 12 in S_4.

10. Prove that the set A_n of all even permutations of degree n forms a finite group of order $\frac{\lfloor n}{2}$ with respect to permutation multiplication.

11. (a) Show that the group $\langle \mathbb{Z}/\langle 8\rangle, +\rangle$ cannot be written as the direct sum of two non-trivial subgroups.

(b) Show that the external direct product of additive group of integers $\mathbb{Z}$ with itself is not a cyclic group.

12. Prove that the number of non-isomorphic abelian groups of order p^n (p prime) is equal to the number of partition of n.

UNIT—IV

13. State and prove Sylow's First Theorem.

14. Show that a group of order $p^2.q$ is solvable, where p and q are prime numbers.

15. (a) Write down all composition series for Q_8.

(b) Show that every group of order 15 is cyclic.

16. Verify class equation for S_3.

UNIT—V

17. (a) In a ring R, $x^3 = x$ for all $x \in R$, then show that R is a commutative ring.

(b) Give an example of an ideal I of a ring R such that I is left ideal but not right ideal.

18. (a) Prove that every finite integral domain is a field.

(b) Prove or disprove there is an integral domain with six elements.

19. (a) Show that $M_2(\mathbb{R})$, the ring of all 2×2 matrices over the field of real numbers is simple.

(b) Find all homomorphism from ring $\mathbb{Z}$ onto $\mathbb{Z}$.

20. (a) If every ideal of a commutative ring R with unity is prime, show that R is a field.

(b) Show that ring $2\mathbb{Z}$ is not isomorphic to ring $5\mathbb{Z}$.

Exam. Code : 211001
Subject Code : 5475

M.Sc. (Mathematics) Ist Semester

MECHANICS—I

Paper—MATH-554

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **TWO** questions from each unit. Each question carries equal marks.

UNIT—I

I. Find the velocity and acceleration of a particle moving along a curve. In the usual notations, show that $\frac{d\vec{t}}{ds} = \kappa \vec{n}$.

II. Determine the components of acceleration of a particle moving along the curve $r = ae^{b\theta}$ such that the radius vector moves with constant angular velocity ω.

III. Define vector angular velocity. In the usual notations show that $\vec{\omega} = \frac{1}{2} \text{curl} \vec{V}$.

IV. If $\frac{d}{dt}$ and $\frac{\partial}{\partial t}$ denote the rate of change relative to fixed frame and moving frame with angular velocity ω respectively, then for any vector $\vec{F}$ show that

$$\frac{d\vec{F}}{dt} = \frac{\partial \vec{F}}{\partial t} + \vec{w} \times \vec{F}$$

and hence find the velocity and acceleration.

UNIT—II

V. A body of mass M, travelling in a straight line, is supplied with power P and is subjected to a resistance Mkv^2, where v is the speed and k is a constant. Prove that the speed of the body cannot exceed a certain value and that, if it starts from rest, it acquires half the maximum speed after travelling a distance $\frac{1}{3k}\log\frac{8}{7}$.

VI. What do you mean by conservative force ? Give example. Show that for a single particle moving in a conservative field of force, the sum of kinetic and potential energy is constant.

VII. A particle of mass m is constrained to execute a rectilinear SHM under a force towards O of magnitude $m\omega^2 x$, x being the particle's displacement from O. When passing through O its velocity is V and when its velocity has become half of V in the same direction, an impulse I is applied to the particle in the direction of its motion. Assuming the same law of force, find the time and total distance travelled from O to the first position of instantaneous rest.

VIII. A particle is projected upward with a velocity V in a medium whose resistance varies as the square of the velocity. Discuss the motion.

UNIT—III

IX. A fixed wire is in the shape of a cardioid $r = a(1 + \cos\theta)$, the initial line being the downward vertical. A small ring of mass m can slide on the wire and is attached to the point r = 0 of the cardioids by an elastic string of natural length 'a' and modulus 4 mg. If the particle is released from rest when the string is horizontal, show that

$$a\dot{\theta}^2(1 + \cos\theta) - g\cos\theta(1 - \cos\theta) = 0.$$

X. A particle is projected with velocity 'u' in a direction inclined at an angle α to the horizontal. Determine the horizontal and vertical displacement after time t on the assumption that gravity is the only force acting. Show that path of trajectory is a parabola.

XI. Discuss the motion of a particle of mass m, moving on the smooth inner surface of the paraboloid of revolution : $x^2 + y^2 = 4az$, whose axis is vertical and vertex downward.

XII. What is a cycloid ? Show that its equation is $s = 4a \sin\psi$ in usual notations. A particle slides down a smooth cycloid whose axis is vertical and vertex downward. Find the velocity of the particle and reaction on it at any point of the cycloid.

UNIT—IV

XIII. Derive the equation of motion of the orbit of a particle moving under central force in terms of reciprocal polar coordinates.

XIV. Show that the inverse square law of force directed towards a fixed point always produces a conic type orbit.

XV. Discuss the motion of a particle moving in an elliptic orbit under the inverse square law of attraction and subjected to a small blow in the tangential direction.

XVI. State Kepler's laws of planetary motion. Two gravitating particles A and B of mass 'm' and 'M' respectively, move under the force of their mutual attraction. If the orbit of A relative to B is a circle of radius 'a' described with velocity v, show that $v = \sqrt{\gamma(M+m)/a}$.

UNIT—V

XVII. Define principal axes a product of inertia. Show that the products of inertia with respect to principal axes are zero.

XVIII. What do you mean by equimomental systems ? State and prove necessary and sufficient conditions for the two systems to be in equimomental.

XIX. Find the moment of inertia of a rigid body about a line having direction cosines $< \lambda, \mu\ \nu >$. Let the rigid body is rotating about this line with angular velocity ω, then find the expression of kinetic energy of the body in terms of its moment of inertia.

XX. State perpendicular axis theorem. Use it to find the moment of inertia of an elliptic disc about a line perpendicular to the plane of the disc.