8-401-2017 Mor

Exam. Code : 211003
Subject Code : 5497

M.Sc. (Mathematics) 3rd Semester

FUNCTIONAL ANALYSIS—I

Paper—MATH-571

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Attempt any **TWO** questions from each unit. All questions carry equal marks.

UNIT—I

1. If M is a closed linear subspace of a Banach space N, show that the quotient space N/M with norm defined by :

$$\| x + M \| = \inf\{\| x + m \| : m \in M\}$$

is a Banach space.

2. Show that a complete subspace of a normed linear space is closed. Give an example of an incomplete normed linear space.

3. If f and g are in L^p with $1 \leq p \leq \infty$, show that $f + g \in L^p$ and $\| f + g \|_p \leq \| f \|_p + \| g \|_p$.

4. Show that a normed linear space X is complete if every absolutely summable series is summable in X.

UNIT—II

5. If a linear transformation T from a normed linear space N into a normed linear space N′ is continuous at a point $x_0 \in N$, show that T is continuous at every point of N.

6. Show that any two norms on a finite dimensional space are equivalent.

7. Show that each closed and bounded subset of a finite dimensional normed linear space is compact. Is the result true for infinite dimensional spaces ? Justify your answer.

8. Show that every locally compact normed linear space is finite dimensional.

UNIT—III

9. Show that a normed linear space N is separable if its conjugate space N* is separable.

10. If x_0 is a non-zero element of a normed linear space N, show that there exists $f_0 \in N^*$ such that $f_0(x_0) = \| x_0 \|$ and $\| f_0 \| = 1$.

11. Show that there exists an isometric isomorphism of a normed linear space N into its second conjugate space N**.

12. Show that every reflexive normed linear space N is a Banach space but converse is not true.

UNIT—IV

13. Show that a one-to-one continuous linear transformation of one Banach space onto another is a homomorphism.

14. If T is a linear transformation from a Banach space B into a Banach space B′, show that T is continuous if and only if graph of T is closed.

15. Show that a non-empty subset X of a normed linear space N is bounded $\Leftrightarrow$ f(X) is a bounded set of numbers for each f in N*.

16. Let B be a Banach space, N a normed linear space and $T_n \in \beta$ (B, N) such that $Tx = \lim T_n(x)$ exists for each $x \in B$, show that T is a continuous linear transformation. If $< T_n(x) >$ is a Cauchy sequence for each $x \in B$, show that $< \| T_n \| >$ is bounded.

UNIT—V

17. Give an example of a Banach space which is not a Hilbert space. Show that every Hilbert space is reflexive.

18. If M is a linear subspace of a Hilbert space, show that $M = M^{\perp\perp} \Leftrightarrow M$ is closed.

19. Show that every orthonormal set in a Hilbert space is linearly independent but converse is not true.

20. Suppose H is a Hilbert space and $y \in H$. Show that there exists $f_y \in H^*$ such that $\| f_y \| = \| y \|$. Show also that given any $f \in H^*$ there exists $y \in H$ such that $f(x) = \langle x, y \rangle$ for all $x \in H$.

Exam. Code : 211003
Subject Code : 5498

M.Sc. Mathematics 3rd Semester

MATH-572 : TOPOLOGY–I

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt **two** questions from each Unit. All questions carry **10** marks each.

UNIT–I

1. Prove that Int (A) = C $\left(\overline{C(A)}\right)$ for any set A, where C(A) denotes the complement of A in X. Further prove that A is open if and only if A = Int (A).
2. Prove that every separable metric space is 2^{nd} countable.
3. Let X and Y be topological spaces and f : X $\rightarrow$ Y a map. Prove that if $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every B $\subset$ Y, then inverse image of each closed set in Y is closed in X.
4. If Y is a space satisfying the second axiom of countability, then prove that every open covering $\{U_\alpha\}$ has a countable subcovering.

UNIT–II

5. (i) Define a connected space and prove that continuous image of a connected space is connected.

   (ii) Prove that a topological space is locally connected if the components of every open subspace of X are open in X.

6. Prove that a topological space is disconnected if and only if there exist a continuous map of X onto the discrete two point space [0, 1].

7. If X is an arbitrary topological space then prove that :
   (i) Each point in X is contained in exactly one component of X.
   (ii) Each connected subspace of X is contained in a component of X.
   (iii) A connected subspace of X which is both open and closed is a component of X.

8. Let $(X, \mathfrak{I})$ be a space and $(Y, \mathfrak{I}_Y)$ a subspace. Then prove that :

$\overline{A}_Y = Y \cap \overline{A}$; $A'_Y = Y \cap A'$; $Y \cap \mathrm{Int}(A) \subset \mathrm{Int}_Y(A)$; $\mathrm{Fr}_Y(A) \subset Y \cap \mathrm{Fr}(A)$.

UNIT–III

9. Let $f : X \to Y$ be closed. Then prove that for all $S \subset X$, $\forall$ open U such that $f^{-1}(S) \subset U$ there exist open V in Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

10. Let $f : X \to Y$ be a homeomorphism. Prove that for any $A \subset X$ the map $g = f|_A : A \to f(A)$ is also a homeomorphism.

11. (i) If f is a continuous mapping of the topological space X into the topological space Y, and $\{x_n\}$ is a sequence of points of X which converges to the point $x \in X$, then the sequence $\{f(x_n)\}$ converges to the point f(x) in Y.
    (ii) Prove that a map $f : X \to Y$ is open if and only if $\forall A \subset X$, $f(A^\circ) \subset (f(A))^\circ$.

12. Let $X = A \cup B$, where A, B are both open or both closed in X. Let $f : X \to Y$ be such that $f|_A$ and $f|_B$ are continuous, then prove that f is also continuous.

UNIT–IV

13. In the product space $\Pi_\alpha Y_\alpha$,
    (i) If S_α is a sub-basis of $(Y_\alpha, \mathfrak{I}_\alpha)$, then prove that the collection $\{<V_\beta> : V_\beta \in S_\beta, \beta \in \Lambda\}$ is a sub-basis for $\Pi_\alpha Y_\alpha$.
    (ii) Let $A_\alpha \subset Y_\alpha$, then A_α has subspace topology on it. Let $\Pi_\alpha A_\alpha$ has product topology $\mathfrak{I}^*$. Considering $\Pi_\alpha A_\alpha$ as a subspace of $\Pi_\alpha Y_\alpha$, $\Pi_\alpha A_\alpha$ has the topology $\mathcal{T}_*$ on it, prove that $\mathfrak{I}^* = \mathcal{T}_*$.

14. (i) Show that infinite product of non trivial discrete spaces is never discrete.
    (ii) Prove that $(\Pi_\alpha A_\alpha)^C = \cup_\alpha (A_\alpha)^C$.

15. Prove that the projection maps are open but they need not be closed.

16. Define the quotient space. Prove that if Y is a quotient space of X and Z is a quotient space of Y then Z is homeomorphic to a quotient space of X.

UNIT–V

17. Show that a closed subspace of a normal space is normal.

18. State and prove Tietz extention theorem.

19. Show that a one to one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism.

20. Prove that $\Pi_\alpha\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ is regular if and only if each Y_α is regular.

2912 (M)

Exam. Code : 211003
Subject Code : 5501

M.Sc. (Mathematics) 3rd Semester

STATISTICS—I

Paper—MATH-577

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Attempt **TEN** questions in all, selecting **TWO** questions from each unit. All questions carry equal marks.

UNIT—I

1. (a) What are the criteria of a good measure of the central tendency ? Also prove that the sum of the squared deviation is least when taken from the mean.

   (b) The average salary of male employees in a firm was Rs. 5,200 and that of females was Rs. 4,200. The mean salary of all the employees was Rs. 5,000. Find the percentage of male and female employees.

2. (a) What do you mean by skewness and kurtosis of a distribution ? Describe briefly their measures.

   (b) The standard deviation of symmetrical distribution is 5. What must be the value of the fourth central moment in order that the distribution be (i) Leptokurtic (ii) Mesokurtic (iii) Platykurtic ?

3. (a) Give the axiomatic definition of probability and describe the additive and subtractive property of its probability function.
   (b) If a fair coin is tossed repeatedly, find the probability of getting m heads before obtaining n tails.
4. (a) Define conditional probability and state and prove Bayes Theorem of probability.
   (b) A player tosses a coin and is to score one point for every head and two points for every tail that turned up. He is to play on until his score reaches or passes n. If p_n is the chance of attaining exactly n score, show that $p_n = \frac{1}{2}(p_{n-1} + p_{n-2})$ and hence find the value of p_n.

UNIT—II

5. (a) What is meant by random variable ? Also distinguish between discrete and continuous random variables and give the example of each type.
   (b) A random process gives measurements x between 0 and 1 with probability density function :

$$f(x) = \begin{cases} 12x^3 - 21x^2 + 10x & , \quad 0 \le x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Find the value of (i) $P\left(x \le \frac{1}{2}\right)$ and $P\left(x > \frac{1}{2}\right)$

(ii) Constant k such that $P(x \le k) = \frac{1}{2}$.

6. (a) Define distribution function of random variable. Also describe a function which satisfies the properties of distribution function and explain.
   (b) The joint probability density function of the two-dimensional random variable (x, y) is given by :

$$f(x, y) = \begin{cases} A(xy + e^x) & , \quad 0 < (x_1, x_2) < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Determine A and examine whether x and y are stochastically independent.

7. (a) What is meant by stochastic independence of random variables ? Also explain n-dimensional random variables.
   (b) Let x_1, x_2 be a random sample of size 2 from a distribution with p.d.f.

$$f(x) = \begin{cases} e^{-x} & , \quad 0 < x < \infty \\ 0 & , \quad \text{otherwise} \end{cases}$$

Show that $y_1 = x_1 + x_2$ and $y_2 = \frac{x_1}{x_1 + x_2}$ are independent.

8. (a) Define marginal and conditional distributions for the case of three dimensional random variables.

(b) The random variables x and y have a joint p.d.f. f(x, y) given by :

$$f(x, y) = \begin{cases} \binom{y}{x} p^x (1-p)^{y-x} \dfrac{e^{-\lambda}\lambda^y}{y!} & ; \quad x = 0, 1, 2, ... ; \; y = 0, 1, 2, ... ; \text{ with } y \geq x, \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find the marginal distribution of x and y. Also determine whether the random variables x and y are independent.

UNIT—III

9. (a) Define mathematical expectation of random variable and describe its properties.

   (b) What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability p of success in each trial ?

10. (a) State and prove Lindeberg-Levy central limit theorem.

    (b) Let random variable x have the distribution :

    $P(x = 0) = P(x = 2) = p$; $P(x = 1) = 1 - 2p$,

    for $0 \leq p \leq \frac{1}{2}$

    For what p is the variance of x maximum ?

11. (a) State and prove Chebyshev's inequality.

    (b) If x is a random variable such that $P_2(x) = 3$ and $P_2(x^2) = 13$, use Chebyshev's inequality to determine a lower bound for $P(-2 < x < 8)$.

12. (a) What is meant by moments ? Also define moment generating function and give its properties.

    (b) Let $\{x_n\}$ be a sequence of independent Bernoulli variables such that $P(x_n = 1) = p_n = 1 - P(x_n = 0)$ for n = 1, 2, 3, ... Show that if $\sum_{n=1}^{\infty} p_n(1-p_n) = \infty$, then the central limit theorem holds for the sequence $\{x_n\}$. What happens if $\sum_{n=1}^{\infty} p_n(1-p_n) < \infty$.

UNIT—IV

13. (a) If random variable x have binomial distribution with parameters n and p, obtain the recurrence relation for its central moments.

    (b) One worker can manufacture 120 articles during a shift, another worker 140 articles, the probabilities of the articles being of a high quality are 0.94 and 0.80 respectively. Determine the most probable number of high quality articles manufactured by each worker.

14. (a) What is hypergeometric distribution ? Find its mean and variance.

    (b) For a geometric distribution with p.m.f. :

    $f(x) = 2^{-x}$; x = 1, 2, 3, ...

    Show that Chebyshev's inequality gives $P\{|x - 2| \leq 2\} > \frac{1}{2}$, while the actual probability is $\frac{15}{16}$.

15. (a) State and prove that geometric distribution lacks memory.

(b) If random variable has a uniform distribution in [0, 1], find the distribution of $y = -2 \log x$. Identify the distribution also.

16. (a) Define the Beta distribution of first and second kind. Also find mean and variance of its first kind.

(b) The random variables x and y are independent, each exponentially distributed with same parameter θ. Find the distribution of $\frac{x}{x+y}$ and identify its distribution.

UNIT—V

17. (a) What do you mean by correlation between the random variables ? How is it measured between two random variables ? How can you use scatter diagram to obtain an idea of the correlation ?

(b) Let $U = ax + by$ and $V = ax - by$, where x, y represent deviations from the means of two measurements on the same individual. If U, V are uncorrelated, show that :

$$\sigma_u \sigma_v = 2\, ab\, \sigma_y \sigma_x (1 - \rho^2)^{1/2}.$$

Here $\sigma_u^2, \sigma_v^2, \sigma_x^2, \sigma_y^2$ and ρ denote the respective variances and correlation coefficient.

(Contd.)

18. (a) Explain partial correlation and multiple correlation. In the usual notations, prove that

$$R_{1.23}^2 = \frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{23}r_{31}}{1 - r_{23}^2} \le r_{12}^2 .$$

(b) The job rating efficiency of an employee seems to be related to the number of weeks of employment. For a random sample of 10 employees, the following data were observed :

Job efficiency (x)	**Weeks of Employment (y)**
55	2
50	4
20	1
55	3
75	5
80	9
90	12
30	2
75	7
70	5

Obtain a regression line of x on y.

19. (a) When are two attributes said to be (i) positively associated and (ii) Negatively associated ? Also define complete association and dissociate of two attributes.

(b) Given that $X = 4Y + 5$ and $Y = kX + 4$ are the lines of regression of X on Y and Y on X respectively, show that $0 < 4k < 1$. If $k = \frac{1}{16}$, find the means of the two variables and coefficient of correlation between them.

20. (a) Define correlation ratio η_{xy} and prove that $1 \geq \eta^2_{xy} \geq r^2$, where r is the correlation coefficient between x and y.

(b) Obtain regression equation of line of Y on X for the following distribution :

$$f(x, y) = \frac{y}{(1+x)^4} e^{-\frac{y}{1+x}} \quad ; x, y \geq 0.$$

Exam. Code : 211003
Subject Code : 5502

M.Sc. Mathematics 3rd Semester

OPERATIONS RESEARCH—I

Paper—MATH-578

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Attempt ***ten*** questions in all, selecting ***two*** questions from each unit. All questions carry equal marks.

UNIT—I

1. Show that the following system of linear equations has degenerate solutions :

$$2x_1 + x_2 - x_3 = 2$$

$$3x_1 + 2x_2 + x_3 = 3$$

2. Use Simplex method to solve the following LPP :

Maximize $z = 2x_1 + 5x_2$

subject to

$$x_1 + 4x_2 \leq 24$$

$$3x_1 + x_2 \leq 21$$

$$x_1 + x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

3. Use Big M-method to

$$\text{Max. } z = 3x_1 - x_2$$

subject to

$$2x_1 + x_2 \le 2$$
$$x_1 + 3x_2 \ge 3$$
$$x_2 \le 4$$
$$x_1, x_2 \ge 0$$

4. Show that if a LPP has a feasible solution then it also has a basic feasible solution.

UNIT—II

5. Let a primal problem be

$$\text{Max. } f(x) = cx \text{ sub. to } Ax \le b,\ x \ge 0$$

and its associated dual be

$$\text{Minimize } g(w) = b^T w \text{ sub. to}$$
$$A^T w \ge C^T,\ w \ge 0$$

If $x_0(w_0)$ is an optimum solution to the primal (dual), then there exists a feasible solution $w_0(x_0)$ to the dual (primal) such that

$$cx_0 = b^T w_0.$$

6. State and prove complementary slackness theorem.

7. Use dual simplex method to

$$\text{Minimize } z = 3x_1 + 2x_2 + x_3 + x_4$$

subject to the constraints :

$$2x_1 + 4x_2 + 5x_3 + x_4 \ge 10$$
$$3x_1 - x_2 + 7x_3 - 2x_4 \ge 2$$
$$5x_1 + 2x_2 + x_3 + 6x_4 \ge 15$$
$$x_1, x_2, x_3, x_4 \ge 0$$

8. Use duality to solve the following LPP :

$$\text{Max. } z = x_1 - x_2 + 3x_3 + 2x_4$$

subject to :

$$x_1 + x_2 \ge -1$$
$$x_1 - 3x_2 - x_3 \le 7$$
$$x_1 + x_3 - 3x_4 = -2$$
$$x_1, x_2, x_3, x_4 \ge 0$$

UNIT—III

9. Explain the Matrix-minima method of obtaining an initial basic feasible solution to a transportation problem.

10. Show that a necessary and sufficient condition for the existence of feasible solution to a transportation problem is $\sum_i a_i = \sum_j b_j, \begin{pmatrix} i=1,2...m \\ j=1,2...n \end{pmatrix}$.

11. Show that the number of basic variables of the general transportation problem at any stage of feasible solution must be $m + n - 1$.

12. Given $x_{13} = 50$, $x_{14} = 20$, $x_{21} = 55$, $x_{31} = 30$, $x_{32} = 35$ and $x_{34} = 25$. Is it an optimal solution to the transportation problem :

					Available units
	6	1	9	3	70
	11	5	2	8	55
	10	12	4	7	90
Reqd. units	85	35	50	45	

If not, modify it to obtain a better feasible solution.

UNIT—IV

13. Give the mathematical formulation of an assignment problem. Explain how to view the problem in terms of an LPP set up.

14. Given below is an assignment problem, write it as a transportation problem :

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{array}{c} A_1 \quad A_2 \quad A_3 \\ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 4 \end{bmatrix} \end{array}$$

15. Solve the following game :

		Player B			
		I	II	III	IV
Player A	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

16. Use matrix oddment method to solve the following 3×3 game :

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

UNIT—V

17. Describe Gomory's method of solving a mixed integer linear programming problem.

18. Use branch and bound technique to solve the following LPP :

Maximize $z = x_1 + 4x_2$

subject to

$2x_1 + 4x_2 \le 7,\ 5x_1 + 3x_2 \le 15$

$x_1, x_2 \ge 0$ and are integers.

19. State the principle of optimality in dynamic programming and give a mathematical formulation of a dynamic programming.

20. Use dynamic programming to solve the LPP :

Maximize $z = 3x_1 + 7x_2$

subject to the constraints :

$x_1 + 4x_2 \le 8$

$x_2 \le 2$

$x_1 \ge 0,\ x_2 \ge 0$