Sr. No. 7676

Exam. Code: 211004
Subject Code : 4953

M.Sc. Mathematics - 4th Sem.

(2517)

Paper: MATH-581: Functional Analysis-II

Time Allowed: 3 hrs. Max. Marks: 100

Note: Attempt TWO questions from each Unit. Each question carries EQUAL marks.

Unit-I

1. Give an example of an infinite dimensional Banach space in which weak convergence of a sequence implied strong convergence. Justify your claim.
2. Show that any closed subspace Y of a normed space X contains the limits of all weakly convergent sequences of its elements.
3. Let $<x_n>$ be a sequence in a normed space X, then:
   a) Strong Convergence implies weak convergence with the same limit.
   b) The Converse of (a) is not generally true.
   c) If Lim $x<\infty$, then weak convergence implies strong convergence.
4. Prove that every bounded sequence in a Hilbert space contains a subsequence which converges weakly.

Unit-II

5. Let $T : H \to H$ be a bounded, linear operator on a Hilbert space H. Then:
   a) If T is self-adjoint, $<Tx, x>$ is real for all $x \in H$.
   b) If H is complex and $<Tx, x>$ is real for all $x \in H$, the operator T is self adjoint.
6. Let the operator $U: H \to H$ be unitary. H is a Hilbert space. Then:
   a) U is isometric i.e. $||Ux||=||x||$ for all $x \in H$.
   b) $||U||=1$ provided $H \neq \{0\}$
   c) U^{-1} is unitary and U is normal.
7. Prove that if P is the projection on a closed linear subspaces M of H, then M reduces an operator T <=> TP=PT.

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8. Give an example of an operator, with justification, which is

   a) self adjoint as well as unitary.

   b) normal but not unitary.

Unit-III

9. If T is an arbitrary operator on H, then the eigen values of T constitute a non empty finite subset of the complex plane. Furthermore, the number of points in this set does not exceed the dimension of the space H.

10. State and prove spectral mapping theorem.

11. Prove that the resolvent set P(T) of a bounded linear operator T on a complex Banach space X is open, hence the spectrum $\sigma(T)$ is closed.

12. Prove that if $X \neq \varphi$ is a complex Banach space and $T \in B(X,X)$ then $\sigma(T) \neq \varphi$.

Unit-IV

13. Let X and Y be normed spaced and $T: X \to Y$ a linear operator. Then T is compact if and only if it maps every bounded sequence (x_n) is X onto a sequence (Tx_n) in Y which has a convergent subsequence.

14. The range R(T) of a compact linear operator $T: X \to Y$ is separable, where X and Y are normed spaces.

15. Let $T: X \to X$ be a compact linear operator on a normed space X. Then for every $\lambda \neq 0$, the range of $T_\lambda = T - \lambda I$ is closed.

16. $T: X \to X$ is a compact linear operator and $S: X \to X$, a bounded linear operator on a normed space X. Then TS and ST are compact.

Unit-V

17. If A is a division algebra, then it equals the set of all scalar multiples of the identity.

18. If the norm in A satisfies the inequality $||xy|| \geq k\, ||x||\, ||y||$ for some positive constant K, then A=0.

19. Show that no topological divisor of zero is invertible.

20. Show that the multiplication in a Banach algebra X is jointly continuous. Also prove that in a Banach algebra X, $||x^n|| \leq ||x||^n$ for all $x \in X$.

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18. Let X be an infinite discrete space. Show that the filterbase $\{A \subset X \mid X - A \text{ is finite}\}$ has no cluster points. Also prove that each filterbase on a space Y has at most one limit point if and only if Y is Hausdorff.

19. Prove that a space is compact if and only if each filter on X has a cluster point in X.

20. Prove that for a filter F on a space X there exists a net P in X such that the filter F is convergent to a point if only if the net P is convergent to the point.

Exam. Code : 211004

Subject Code : 4954

M.Sc. Mathematics 4th Semester

TOPOLOGY—II

Paper—MATH-582

Time Allowed—Three Hours] [Maximum Marks—100

Note :— Attempt ***two*** questions from each unit. All questions carry **10** marks each.

UNIT—I

1. Prove that a space is completely normal if and only if it is hereditarily normal.

2. Prove that every metric space is T_5. Is every subspace of a T_4 space completely regular ? Justify. Give an example of a normal space which is not completely normal.

3. Prove that the collection $\{f^{-1}(0,1] \mid f : X \to I = [0, 1],$ f is continuous$\}$ is a base for the topology on X if and only if X is completely regular.

4. Prove that every compact Hausdorff space is normal. Give an example of a normal space which is not Hausdorff.

UNIT—II

5. Prove that any subset of the real line is compact if and only if it is closed and bounded.

6. Let X be an infinite set with cofinite topology. Is it compact ? Is it normal ? Justify your answers. Prove that every space is subspace of some normal space.

7. Let $p : \mathbf{Z}^+ \to X$ be a sequence converging to x_0. Prove that the set $\{x_0\} \cup \{p(\mathbf{Z}^+)\}$ is compact. Also prove that any infinite subset of A not containing the point x_0 is not compact.

8. Let $f : X \to Y$ where X is Hausdorff and Y is compact. Prove that f is continuous if and only if the graph $G(f) = \{(x, f(x)) \mid x \in X\}$ is a closed subset of X x Y.

UNIT—III

9. Let X be Hausdorff. Prove that any countable open cover of X has a finite subcover if and only if every sequence in X has a cluster point.

10. Prove that every locally compact Hausdorff space is Tichonov.

11. Prove that every point of a Hausdorff space X has neighborhood with compact closure if and only if whenever $K \subset U$ where K is compact and U is open, then there exists an open V with compact closure such that $K \subset V \subset cl(V) \subset U$, where cl(V) denotes the closure of V.

12. Prove that the one point compactfication of the space X is Hausdorff if and only if X is locally compact and Hausdorff.

UNIT—IV

13. Prove that a space is Tichonov if and only if it is homeomorphic to a subspace of some compact Hausdorff space.

14. What is the Stone Cech compactfication $(\beta(X), \rho)$ for a Tichonov space X ? Prove that for any continuous function $f : X \to Y$, where X is Tichonov and Y is compact, there exists a continuous function $F : \beta(X) \to Y$ such that $F \,\square\, \rho = f$.

15. Prove that $\beta(X)$ is the largest compactification of a Tichonov space X in the sense that any other compactification of X is a quotient space of $\beta(X)$.

16. Prove that a completely regular space is connected if and only if its Stone Cech compactification is connected.

UNIT—V

17. Let **B** be a filterbase consisting of single subset A of X. Justify the following statements :

    (a) If A is singleton set {x} then **B** converges to x.

    (b) If A consists of more than one point then **B** clusters at each point of closure of A.

    Give an example of a filterbase with cluster points but no limit.

UNIT—V

17. If x_1, y_1 is the fundamental solution of $x^2 - dy^2 = 1$, then prove that any positive solution of the equation is given by x_n, y_n, where x_n, y_n are the integers determined from

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n, \ n = 1, 2, 3 \ldots$$ 10

18. Prove that the value of any infinite continued fraction is an irrational number. Determine the unique irrational number represented by infinite continued fraction [3; 6, 1, 4, 1, 4 ....]. 10

19. (a) If $C_k = \dfrac{p_k}{q_k}$ is the k^{th} convergent of simple continued fraction $[a_0; a_1, a_2 \ldots, an]$, then for

$$1 \le k \le n, \ p_k q_{k-1} - q_k p_{k-1} = (-1)^k.$$ 5

(b) Solve $172x + 20y = 1000$ by means of simple continued fraction. 5

20. Prove that the convergents with even subscripts of a simple continued fraction forms a strictly increasing sequence whereas the convergents with odd subscripts forms strictly decreasing sequence. Moreover show that the convergents with an even subscript is less than that with an odd subscript. 10

Centre No → 3 Morning 27/05/17

Exam. Code : 211004
Subject Code : 4961

M.Sc. (Mathematics) Semester—IV

MATH–586 : NUMBER THEORY

Time Allowed—3 Hours] [Maximum Marks—100

Note :— The candidates are required to attempt **TWO** questions from each Unit. Each question carries equal marks.

UNIT—I

1. (a) Solve the simultaneous congruence $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$. 5

(b) Find three consecutive integers, each having a square factor. 5

2. Let r be a primitive root of the odd prime p. Prove that :

(i) $r^{\frac{p-1}{2}} \equiv -1 \pmod p$

(ii) If s is any other primitive root of p, then r s is not a primitive roof of p. 10

3. State and prove Wilson's theorem. 10

4. Prove that if p is an odd prime then for $k \ge 1$, there exists a primitive root for p^k. Further prove that there do exist primitive roots for $2p^k$ also. 10